The Incomplete Gamma Function
Part I - Derivation and Solution

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In this white paper we will derive the solution to the incomplete gamma function. To that end we will use the following hypothetical problem...

Our Hypothetical Problem

Given the following incomplete gamma function (as defined below) parameters...

$$\alpha = 0.50 \text{ ...and... } x = 1.20 \tag{1}$$

Answer the following questions...

Question 1: What is the value of the upper incomplete gamma function given the parameters above?

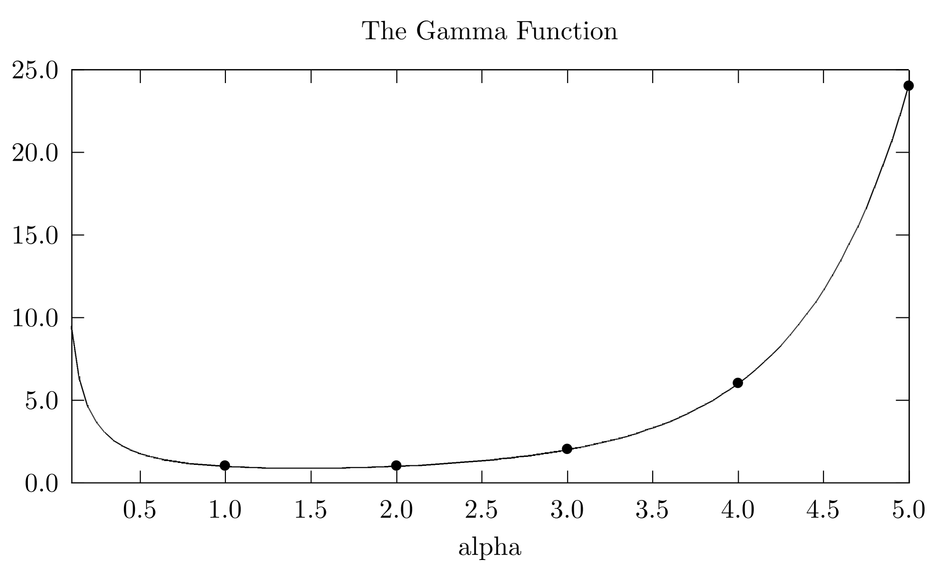
Question 2: Using the result from the question above what is the approximate change in value given that we increase x from 1.20 to 1.21?

The Mathematics

We will define the function $\Gamma(\alpha)$ to be the standard gamma function, which has a lower integral bound of zero and an upper integral bound of infinity. The equation for the standard gamma function is... [1]

$$\Gamma(\alpha) = \int\limits_0^\infty u^{\alpha-1} \operatorname{Exp}\left\{-u\right\} \delta u \text{ ...where... } \alpha > 0 \tag{2}$$

The following graph charts Equation (2) above over the real number interval $[\alpha > 0, \alpha \leq 5]$...



Note that the excel function for the gamma function is...

$$\Gamma(\alpha) = \text{EXP(GAMMALN(alpha))} \tag{3}$$

We will define the function $\Gamma(\alpha, x)$ to be the upper incomplete gamma function, which is the standard gamma function as defined by Equation (2) above but with a lower integral bound of $x > 0$. The equation for the upper incomplete gamma function is...

$$\Gamma(\alpha, x) = \int\limits_x^\infty u^{\alpha-1}\,\text{Exp}\left\{-u\right\}\delta u \;\; \text{...where...} \;\; \alpha > 0 \;\; \text{...and...} \;\; x > 0 \tag{4}$$

We will define the function $\gamma(\alpha, x)$ to be the lower incomplete gamma function, which is the standard gamma function as defined by Equation (2) above but with an upper integral bound of $x > 0$. The equation for the lower incomplete gamma function is...

$$\gamma(\alpha, x) = \int\limits_0^x u^{\alpha-1}\,\text{Exp}\left\{-u\right\}\delta u \;\; \text{...where...} \;\; \alpha > 0 \;\; \text{...and...} \;\; x > 0 \tag{5}$$

The relationship between Equations (2), (4) and (5) above is...

$$\Gamma(\alpha) = \gamma(\alpha, x) + \Gamma(\alpha, x) \tag{6}$$

Note that when we set the variable α in Equation (4) above to zero then the upper incomplete gamma function becomes the exponential integral, which means that the exponential integral is a special case of the upper incomplete gamma function. The equation for the exponential integral is... [3]

$$E_1(x) = \lim_{\alpha \to 0} \Gamma(\alpha, x) = \int\limits_x^\infty u^{-1}\,\text{Exp}\left\{-u\right\}\delta u \tag{7}$$

We will define the function $\Omega(\alpha, x)$ to be the gamma distribution. The equation for the gamma distribution where the variable θ is the rate parameter is... [2]

$$\Omega(\alpha, x) = \int\limits_0^x \frac{u^{\alpha-1}\,\text{Exp}\left\{-u\,\theta^{-1}\right\}}{\theta^\alpha \Gamma(\alpha)}\,\delta u \;\; \text{...where the excel function is...} \;\; \text{GAMMA.DIST(x,alpha,theta,true)} \tag{8}$$

We will define the function $\Omega_s(\alpha, x)$ to be the standard gamma distribution. When we set the rate parameter θ in Equation (8) above equal to one then that distribution becomes the standard gamma distribution as defined below...

$$\Omega_s(\alpha, x) = \int\limits_0^x \frac{u^{\alpha-1}\,\text{Exp}\left\{-u\right\}}{\Gamma(\alpha)}\,\delta u \;\; \text{...where the excel function is...} \;\; \text{GAMMA.DIST(x,alpha,1,true)} \tag{9}$$

Using Equations (2) and (9) above we can rewrite the lower incomplete gamma function (Equation (5)) above as...

$$\gamma(\alpha, x) = \int\limits_0^x u^{\alpha-1}\,\text{Exp}\left\{-u\right\}\delta u = \Omega_s(\alpha, x) \times \Gamma(\alpha) \;\; \text{...where...} \;\; \alpha > 0 \;\; \text{...and...} \;\; x > 0 \tag{10}$$

Using Equations (3) and (9) above we can rewrite Equation (10) above as...

$$\gamma(\alpha, x) = \text{GAMMA.DIST(x,alpha,1,true)} \times \text{EXP(GAMMALN(alpha))} \tag{11}$$

Using Equations (6) and (11) above we can rewrite the upper incomplete gamma function (Equation (4)) above as...

$$\Gamma(\alpha, x) = \Gamma(\alpha) - \gamma(\alpha, x) = \text{EXP(GAMMALN(alpha))} \times (1 - \text{GAMMA.DIST(x,alpha,1,true)}) \tag{12}$$

The Derivatives

Using Equation (4) above the equation for the upper incomplete gamma function is...

$$\Gamma(\alpha, x) = \int\limits_{x}^{\infty} u^{\alpha-1}\,\mathrm{Exp}\left\{-u\right\}\delta u \tag{13}$$

We will make the following change of variables so as to reverse the bounds of integration...

$$v = -u \;\text{ ...where... }\; \frac{\delta v}{\delta u} = -1 \;\text{ ...such that... }\; \delta u = -\delta v \tag{14}$$

Using the definitions in Equation (14) above we can rewrite Equation (13) above as...

$$\Gamma(\alpha, x) = \int\limits_{-x}^{-\infty} -v^{\alpha-1}\,\mathrm{Exp}\left\{v\right\}(-\delta v) = -\int\limits_{-\infty}^{-x} -v^{\alpha-1}\,\mathrm{Exp}\left\{v\right\}\delta v$$

Using Equations (13), (14) and (15) above the derivative of the upper incomplete gamma function is...

$$\frac{\delta}{\delta x}\Gamma(\alpha, x) = -\left(x^{\alpha-1}\,\mathrm{Exp}\left\{-x\right\}\right) \;\text{ ...which is the negative of the integrand} \tag{15}$$

Using Equation (5) above the equation for the lower incomplete gamma function is...

$$\gamma(\alpha, x) = \int\limits_{0}^{x} u^{\alpha-1}\,\mathrm{Exp}\left\{-u\right\}\delta u \tag{16}$$

Using Equation (16) above the derivative of the lower incomplete gamma function is...

$$\frac{\delta}{\delta x}\gamma(\alpha, x) = x^{\alpha-1}\,\mathrm{Exp}\left\{-x\right\} \;\text{ ...which is the integrand itself} \tag{17}$$

The Answers To Our Hypothetical Problem

Question 1: What is the value of the upper incomplete gamma function?

Using Equation (12) above and the parameters to our hypothetical problem the answer to Question 1 is...

$$\begin{aligned}\Gamma(0.50, 1.20) &= \mathrm{EXP(GAMMALN(0.50))} \times (1 - \mathrm{GAMMA.DIST(1.20{,}0.50{,}1{,}true)})\\ &= 1.77245 \times (1 - 0.87866)\\ &= 0.215061\end{aligned} \tag{18}$$

Question 2: What is the approximate change in value given that we change x from 1.20 to 1.21?

Using Equation (12) above and the revised parameters the answer to Question 1 is...

$$\begin{aligned}\Gamma(0.50, 1.21) &= \mathrm{EXP(GAMMALN(0.50))} \times (1 - \mathrm{GAMMA.DIST(1.21{,}0.50{,}1{,}true)})\\ &= 1.77245 \times (1 - 0.88021)\\ &= 0.212331\end{aligned} \tag{19}$$

Using Equation (15) above and the revised parameters to our problem the derivative of the equation in Question 1 above with respect to the lower bound is...

$$\frac{\delta}{\delta x}\Gamma(\alpha, x) = -\left(1.2000^{0.5000-1} \times \mathrm{Exp}\left\{-1.2000\right\}\right) = -0.274951 \tag{20}$$

Using Equations (18), (19) and (20) above the answer to Question 2 is...

$$\begin{aligned}\text{Actual change} &= 0.212331 - 0.215061 = -0.002730\\ \text{Approximate change} &= -0.274951 * (1.21 - 1.20) = -0.002750\end{aligned} \tag{21}$$

References

[1] Gary Schurman, *The Gamma Function,* April, 2016

[2] Gary Schurman, *The Gamma Distribution,* April, 2016

[3] Gary Schurman, *The Exponential Integral,* November, 2017